

Vortices on closed surfaces

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Abstract

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We consider N point vortices s_j of strengths κ_j moving on a closed (compact, boundaryless, orientable) surface S with riemannian metric g . As far as we know, only the sphere or surfaces of revolution, the latter qualitatively, have been treated in the available literature. The aim of this note is to present an intrinsic geometric formulation for the general case. Since the pioneer works of Bogomolov [8] and Kimura/Okamoto [36] on the sphere S^2 , it is known that stream function produced by a unit point vortex at $s_o \in S$ on a background uniform counter vorticity field is given by Green's function $G_g(s, s_o)$ of the Laplace-Beltrami operator $\Delta_g = \text{div} g \circ \text{grad}_g$. It behaves as $\log d(s, s_o)/2\pi$ near s_o . Desingularizing the stream function G_g is therefore mainstream mathematics (no pun intended). An energy core argument shows that a single vortex s_o drifts according to the Hamiltonian system (Ω_g, R_g) , where Ω_g is the area form of g and R_g is Robins's function $R_g(s_o) = \lim_{s \rightarrow s_o} G_g(s, s_o) - \log d(s, s_o)/2\pi$. If S has genus zero, it is known that $R_g = \Delta_g^{-1} K + \text{trace} \Delta_g^{-1} / A(S)$. The collective Hamiltonian keeps the same form of C.C.Lin's now classical paper [45], with

$$\Omega_{\text{collective}} = \sum_{j=1}^N \kappa_j \Omega(s_j) \ , \ H = \sum_{1 \leq i < j \leq N} \kappa_i \kappa_j G_g(s_i, s_j) + \sum_{\ell=1}^N \frac{1}{2} \kappa_\ell^2 R_g(s_\ell) \ .$$

For Jordan domains $D \subset S$ the structure is the same, using the appropriate hydrodynamical Green function. The extension for vortices with mass is also immediate. Under conformal changes of metrics $\tilde{g} = h^2 g$ the symplectic form changes accordingly to $\Omega_{\tilde{g}} = h^2 \Omega_g$ while the new Hamiltonian is given by

$$\tilde{H} = H - \frac{1}{4\pi} \sum_{\ell=1}^N \kappa_\ell^2 \log(h(s_\ell)) - \frac{\kappa}{\tilde{A}(S)} \sum_{\ell=1}^N \kappa_\ell \Delta^{-1} h^2(s_\ell) \ , \ \kappa = \sum_{\ell=1}^N \kappa_\ell \ .$$

The presence of the total vorticity in the conformal change formula reflects the fact that when the sum of the vorticities vanish, the collective vortex stream function $\psi(s; s_1, \dots, s_N)$ is independent of the conformal metric $\tilde{g} = \exp(2\phi)g$. In this case the \tilde{g} -regularization of ψ at any of the vortices simplifies: one just subtracts off $\frac{1}{2\pi}\phi(s_j)$ from the g -regularized stream function at s_j . In particular, when S is conformal to the standard sphere, making an artificial puncture at any given $s^* \in S$ allows to easily write vortex motions for any metric on the sphere. We give a simple proof of Kimura's conjecture that a dipole describes geodesic motion. Searching for integrable vortex pairs systems on Liouville surfaces is in order. The vortex pair system on a triaxial ellipsoid extends Jacobi's geodesics. Is it Arnold-Liouville integrable? Not in our wildest dreams is another possibility: that quantizing a vortex system could relate with a million dollars worth question, but we took courage - nerve is more like it - to also present it.

1 Introduction

The purpose of this note is to formulate vortex motions in intrinsic fashion. The (now classical) results of C.C.Lin ([45], 1941) are generalized and reinterpreted. Complex coordinates $z \in \mathbf{C}$ will not be used (well, almost). The key observation is that the core energy desingularization procedure ([27], 1997) geometrizes in a nice way, matching perfectly with Green's function behavior.

1.1 Vortex history on a capsule

Helmholtz' "Wirbel" paper, short for (*Über integrale der hydrodynamischen gleichungen welche den Wirbelbewegungen entsprechen*, [31], 1858) came out a century+ after Euler's *Principes généraux du mouvement des fluides* ([25], 1757), where the mathematical definition of vorticity first appeared¹. Eighteen years later Kirchhoff presented the equations for point vortices in the plane (*Vorlesungen*, [37], chapter XX, p. 259, eq.(14)):

$$m_j \frac{dx_1}{dt} = \frac{\partial P}{\partial y_j}, \quad m_j \frac{\partial y_1}{\partial t} = -\frac{\partial P}{\partial x_j}, \quad P = \sum \frac{1}{\pi} m_i m_j \log \rho_{ij}$$

(nowadays vortex strengths are denoted by κ or $\Gamma/2\pi$ instead of m , $1/\pi$ is omitted, and H replaces P).

This is a Hamiltonian system (Ω, P) , where the symplectic form is the combination of planar areas weighted by the vorticities, $\Omega_{system} = \sum_j \kappa_j dx_j \wedge dy_j$. Only in 1941 the correct Hamiltonian was found by C.C.Lin, [45], for the case of arbitrary planar domains $D \subset \mathbf{C}$. See [49] and [7] for treatises and [1], [2], [6] for recent reviews.

1.2 Vortices on curved surfaces

Vortex dynamics on the sphere and hyperbolic plane started with Bogomolov [8] and Kimura and Okamoto [35]. Basically, since the late 1970's up to now only the constant curvature case has been considered. Follows a very incomplete list (we apologize for omissions): Kimura and Okamoto ([36], 1987), Pekarski and Marsden ([52], 1998), Kidambi and Newton ([33], 1998, [34], 2000), Lim et al. ([44], 2001), Cabral and Boatto ([15], 2003), Borisov and al. ([10], [11], 1998, [12], 2000), Laurent-Polz ([42], 2005), Tronin ([61], 2006). The case of surfaces of revolution has just been touched recently, in qualitative terms ([57], 2002) or perturbatively ([16], 2007).

How about the general case? Vortex equations in the case of domains conformal to a planar domain were obtained by Hally [28]. The key aspect is how to desingularize the *velocity* field at a vortex. Using isothermic coordinates, and after some heuristics (complicated for mathematicians) he found additional "self-terms" attributed to the *variations in the surface curvature* near each vortex. For a surface homeomorphic to the z -plane \mathbf{C} , with metric $g = h^2(z, \bar{z}) dz \otimes d\bar{z}$, Hally's equations are

$$h^2(z_j, \bar{z}_j) \dot{\bar{z}}_j = i \sum_{\ell, \ell \neq j} \frac{\kappa_\ell}{z_\ell - z_j} + i \kappa_j \frac{\partial}{\partial z_j} \log(h(z_j, \bar{z}_j)) \quad , \quad (1 \leq j \leq N) \quad . \quad (1.1)$$

A single unit vortex obeys the "self-motion equation"

$$\dot{\bar{z}} = i h^{-2}(z, \bar{z}) \frac{\partial}{\partial z} \log(h(z, \bar{z})) \quad . \quad (1.2)$$

We provide corrections to [28] for the case of nonzero total vorticity. Our results are valid for closed surfaces of arbitrary genus and Jordan domains on them.

¹Circa 340 B.D. Aristoteles tried to explain typhoons in his *Metereologica*. The importance of vorticity in microfluid motion was clearly recognized by Leonardo da Vinci. Decartes envisaged vortices as powerhouses for planetary movement. W. and J.J. Thomson used vortices for an atomic theory, discarded after Rutherford's 1912 experiments. Some say string theory is a souped up version of the primeval vortex soup.

2 Laplace-Beltrami operator and 2-dimensional hydrodynamics

2.1 Complex and symplectic geometry

Hereafter S will always denote a two dimensional compact boundaryless orientable manifold (hereafter called closed), endowed with a riemannian metric $g = \langle \cdot, \cdot \rangle$, and $D \subset S$ a Jordan domain (a region with compact closure on a whose boundary consists of a finite number of connected components each diffeomorphic to a circle). The underlying *Riemann surface structure* for S is given by the atlas \mathcal{A} of Laplace-Beltrami isothermic coordinates. *Complex geometry* keeps the notion of angle between tangent vectors u, v (thus the rotation operator J by 90 degrees of tangent vectors is well defined), but neglect lenghts $|u|$ and areas $\Omega(u, v)$. *Symplectic geometry*, on the other hand, keeps the area form but neglects the angles. Merging different symplectic forms with the same complex geometry produces conformal riemannian metrics. In 2-dimensions all metrics are automatically Kahler.

Functions (usually velocity potentials or stream functions) produce vectorfields under the gradient and symplectic gradient operators grad (usually denoted ∇), $\text{sgrad} := J \circ \text{grad}$ (also denoted ∇^\perp),

$$d\phi(\bullet) = \langle \text{grad}\phi, \bullet \rangle = \Omega(\text{sgrad}(\phi), \bullet) . \quad (2.1)$$

There is a “baby Hodge theory” in two dimensions: one identifies functions with two-forms via $f \leftrightarrow f\Omega$ and vectorfields with 1-forms via $v \leftrightarrow \omega = \langle v, \bullet \rangle$. Let C be a closed curve enclosing a domain homeomorphic to a disk $D \subset S$. Orient C such that the frame n, t is positive, where t is the unit tangent vector and n the exterior normal. Let v be a vectorfield. The familiar formula

$$\oint_C \langle v, n \rangle ds = \int \int_D \text{div}(v) \Omega \quad (2.2)$$

makes perfect sense. The coordinate free definition of the *divergence operator*, sending vectorfields to functions, can be given as follows (applying Stokes’ theorem and looking at the integrands):

$$d\langle J(v), \bullet \rangle = \text{div}(v) \Omega . \quad (2.3)$$

Incompressible hydrodynamics means: $\text{div}(v) = 0$ except for sources or sinks. On a compact surface S their divergences must add to zero. In *potential* flow (and always incompressible), $v = -\text{grad}\phi$ for a velocity potential ϕ which in general is not globally defined (it is a multivalued function in the Riemann surface sense).

Define the Laplace operator by $\Delta = \text{div} \circ \text{grad}$. It is a negative definite self-adjoint operator with respect to $\langle f, g \rangle = \int_M fg \Omega$. Incompressibility implies $\Delta\phi = 0$ (up to singularities). Thus the velocity potential $\phi(s), s \in S$ is harmonic up to singularities, and complex function theory of S (viewed as a Riemann surface) comes to fore. The conjugate harmonic function ψ is called the *stream* function. Combined they form the *velocity potential* $F = \phi + i\psi$ (in planar hydrodynamics, $u + iv = -\overline{F'(z)}$). Clearly v is tangent to the level lines of ψ , and can be described symplectically from ψ via

$$d\psi = \Omega(v, \bullet) \Leftrightarrow v = \text{sgrad}(\psi) = -\text{grad}(\phi) . \quad (2.4)$$

The *circulation* $\oint_C \langle v, t \rangle ds$ of v around a closed curve $C = \partial D$ is an object of fundamental importance. Using Stokes’ theorem it can be transformed into a double integral,

$$\oint_C \langle v, t \rangle ds = \oint_C \langle J(v), J(t) \rangle ds = \oint_C \langle J(v), -n \rangle ds = - \int \int_D \text{div}(Jv) \Omega$$

But $J(v) = J(J\text{grad}\psi) = -\text{grad}\psi$. Hence

$$\oint_C \langle v, t \rangle ds = \int \int_D \Delta\psi \Omega \quad (2.5)$$

A coordinate free definition of the surface *rotational* or *vorticity* $\omega = \text{rot}v$ is then

$$d\langle v, \bullet \rangle = \text{rot}(v) \Omega . \quad (2.6)$$

In passing, we mention that discrete versions of all these objects have been proved useful in computer graphics and flow visualization, see e.g. [24], [53].

2.2 Poisson's equation and Green functions for the Laplace operator

The vorticity distribution ω forms “the sinews and muscles of fluid motion” (Küchemann, [43], 1965). No wonder 2-dimensional hydrodynamics is governed by Poisson's equation (inverting the Laplace operator). Perhaps the noblest equation of them all, it is the holy grail of every mathematician, pure or applied:

$$\Delta\psi = \omega, \quad \omega \in \text{some class of functions} . \quad (2.7)$$

On a closed surface the source term must average to zero:

$$\int \int_S \omega \Omega = 0 . \quad (2.8)$$

[Proof. $\int \int_S \omega \Omega = \int \int_S \Delta\psi dS = \int \int_S \nabla \cdot \nabla\psi dS = \oint_{\partial S} \langle \nabla\psi, n \rangle ds = 0$, since there is no boundary. Alternatively, take a small curve C enclosing a disk D . The circulation of the velocity field can be computed as the double integral of the vorticity on D and minus that integral on $S - D$.]

In two dimensions vorticity is a “material property”, $L_v\omega = 0$, and this translates into

$$\frac{\partial\omega}{\partial t} + \langle v, \text{grad}\omega \rangle = 0, \quad v = \text{sgrad}\psi . \quad (2.9)$$

Choosing an appropriate class of vortex functions is an important mathematical problem. In many concrete examples, the “vorticity ountour map” has sharply defined islands of high vorticity (positive or negative) in a sea of roughly constant (often zero) vorticity. In time, such regions can blend or develop filaments. The evolution of the system (2.7, 2.9) may reach a stand-off by the appearance of singularities.

At any rate, given $\kappa_j, 1 \leq j \leq N$, take geodesics disks D_j of equal but very small area $\text{Area}(D_j) = \epsilon^2$ centered at s_j and define

$$\omega(s) = \frac{\kappa_j}{\epsilon^2}, \quad s \in D_j \quad (2.10)$$

and the suitable constant elsewhere. One can prove that an initial state consisting of small nearly circular vortex cores remains stable.

A *vortex pair* is the case $N = 2$ with opposite vorticities. Their ubiquitous presence in fluid flows is fascinating. In section 4.3 we give a simple proof of a conjecture by Kimura [35]: a dipole describes a geodesic on S (zooming across the fluid very quickly).

A *monovortex* consists on a single vorticity island on a sea of distributed counter vorticity (uniformly, say) so that the average vorticity is zero. By linearity, any system of vortex patches can be decomposed on a weighted (positive and negative) sum of monovortices. The following *gedanken* experiment may help. Consider a fluid at rest occupying the closed surface S . All of a sudden, stir around a given point s_o , producing a vortex singularity there. What would be the resulting flow? Since there is no distinguished point s_* to be taken as a counter-vortex, the opposite vorticity should be distributed as homogeneously as possible on the surface. [*Vorticity* is an *area* related concept (via Hodge theory, vorticity is thought as a two-form), and hence the area element of S should be used to perform this homogenization.] Note that a distinguished s_* may exist if S has symmetries, for instance if s_o is the south pole of a surface of revolution; then s_* is the north-pole; those situations would be quite unstable, though².

Point vortex systems are meant as finite dimensional ODE aproximations for Euler's equations (geodesic flow on the group of area preserving transformations of S) or the equivalent PDEs (2.7,2.9), assuming concentrated vorticities at points $s_j(t), 1 \leq j \leq N$. As $\epsilon \rightarrow 0$, how to obtain a suitable ODE approximation for the PDE system (2.7, 2.9)? This is done in two steps. Firstly, one needs the stream function for a marker particle on a flow generated by bound point vortices. The second step is a desingularization procedure in order to remove the influence of each vortex on itself. Then we let them “dance” under each other's influence and its own drift.

The first step is achieved with Green functions od the Laplace-Beltrami operator.

²The lack of unicity in some Euler flows is a known possibility; Joe Keller's “tea pot effect” is a sometimes dramatic example [32].

3 Point vortex equations

Definition 1 $G = G_{(S,g)}$ is the Green function of Δ_g ,

$$\Delta^{-1}\omega(s) = \int_M G(s,r)\omega(r)dA(r), \quad \Delta^{-1}\Delta\omega = \omega - \frac{1}{A} \int_S \omega dA. \quad (3.1)$$

$G(s, s_o)$ is a smooth function outside the diagonal, characterized by the following properties [50]:

$$\Delta G(s, s_o) = -\frac{1}{\text{Area}(S)} \quad (s \neq s_o), \quad G(s, s_o) - \log d(s, s_o)/2\pi \text{ bounded}, \quad \int_S G(p, q)\Omega(q) = 0, \quad G(s, s_o) = G(s_o, s). \quad (3.2)$$

Symmetry is to be expected from this formal calculation:

$$G_{s_o}(s) = \int \int_S G_{s_o}(s_1)\Delta G_s(s_1)ds_1 = \int \int \nabla G_{s_o} \cdot \nabla G_s - \int_{\partial S} G_{s_o}\partial_n G_s = \int \int \nabla G_{s_o} \cdot \nabla G_s.$$

Hydrodynamically, $\psi = G(s, s_o)$ fulfills all the requisites to be the stream function of a unit monovortex at s_o . In the electrostatic interpretation, $G(p, q)$ is the potential at p of a positive unit point charge at q , on a conductor with total charge zero (the background negative charge is uniformly distributed). A test particle s with positive but negligible charge on the field generated by a unit charge will move according to

$$\dot{s} = \text{grad}G(s; s_o), \quad s \neq s_o. \quad (3.3)$$

while a fluid particle (marker) s on the flow generated by an unit strength bound vortex s_o will move according to

$$\dot{s} = \text{sgrad}G(s; s_o), \quad s \neq s_o. \quad (3.4)$$

3.1 Eppure si muove: vortex drift under its own stream

Newton's third law precludes the action of the material particle at s_o on itself, but the rest of the universe $S - s_o$ conspires to impinge a "collective reaction" on s_o and makes it drift. In order to capture this motion, a desingularization procedure is needed. To make that long story short, a single vortex will move according to

Prescription 1

$$\dot{s}_o = \text{sgrad}R(s_o), \quad R(s_o) = \lim_{s \rightarrow s_o} G(s, s_o) - \frac{\log d(s, s_o)}{2\pi} \quad (3.5)$$

where d is the geodesic distance between points as measured with g .

The explanation for this rule is given in Proposition 1 below. We were filled with a religious feeling by the fact that the regular part R is a notable object from the geometric function theory of the Laplace-Beltrami operator, known as Robin's function of g ([51], [51]). Vortex drift belongs to mainstream mathematics!

Remark 1 Jean Steiner [59] has shown that for any metric g on S^2

$$R_g(s) - \frac{1}{2\pi} (\Delta_g^{-1} K_g)(s) = \frac{1}{A_g(S)} \text{trace} \Delta_g^{-1} \quad (3.6)$$

where K is the Gaussian curvature. For higher genus the RHS is no longer a constant and its fluctuation does not have a simple geometric interpretation.

Remark 2 Marker's equation (3.4) becomes a 1 1/2 degrees of freedom system, by inserting $s_o(t)$, a solution of (3.5). Chaotic marker equations are possible even for one vortex systems.

A basic observation in this paper is that the core energy argument in [27] geometrizes nicely:

Proposition 1 *The desingularization rule (3.5) encodes the core energy renormalization argument.*

Proof. Consider a small Gauss system of normal coordinates around s_o . The gradient $\text{gradd}(s, s_o)$ is a unit vector along the geodesic rays. The symplectic gradient $\text{sgrad } d(s, s_o)$ is obtained by composition with J and is therefore tangent to the geodesic circles. Its flow leaves the geodesic disks invariant, and the same is true for any function of the distance, in particular $\psi_{\text{sing}} = \log d(s, s_o)$. For $\log d(s, s_o)$ the vectors rotate with speed inversely proportional to $d(s, s_o)$ on a geodesic circle. A simple back of the envelope reasoning shows that the kinetic energy confined in this blob is logarithmically infinite. It can be removed (in physics jargon, renormalized) provided only a small quantity of kinetic energy crosses the boundary of a geodesic disk in finite time. We invoke [27]: energy diffusion “can be neglected in the limit as the radius of the ball tends to zero” , section 5). This is true precisely due to the the fact that behavior of Greens’s function $G(s, s_o) - d(s, s_o)/2\pi$ is bounded.

3.2 Collective vortex motions: Hamiltonian description

Collective motions are obtained under

Prescription 2 *Each vortex moves on the stream generated by all the others together with its own drift term.*

Rewriting this prescription in Hamiltonian form is straightforward. In view of the symmetry properties of Green’s functions, C.C.Lin’s [45] arguments geometrize directly.

Theorem 1 *The dynamics of N point vortices s_j of strenghts κ_j moving on S is a Hamiltonian system on $S \times \dots \times S$ with symplectic form*

$$\Omega_{\text{collective}} = \sum_{1 \leq j \leq N} \kappa_j \Omega(s_j) \quad (3.7)$$

and Hamiltonian

$$H = \sum_{i < j} \kappa_i \kappa_j G(s_i, s_j) + \sum_{i=1}^N \frac{1}{2} \kappa_i^2 R(s_i) . \quad (3.8)$$

where the self interaction terms are given by Robin’s function

$$R(s) = \lim_{r \rightarrow s} G(r, s) - \frac{\log(d(r, s))}{2\pi} . \quad (3.9)$$

Remark 3

- i) *The key aspect for deriving vortex equations is the geometric desingularization prescription (3.9). As we just observed, it can be solidly justified via the core energy method, see section 5 of Flucher and Gustafsson ([27], 1997).*
- ii) *We omitted the possibility of collisions. See [29], [30] for collisions and regularizations for 3 and 4 vortex motions on the plane.*

There is an immediate extension to vortices with mass that may be relevant due to current interest in Bose-Einstein condensates, see [40], [13], [54]). These systems will exhibit slow-fast Hamiltonian phenomena [48].

Corollary 1 *Let each s_j have a mass m_j besides its vorticity κ_j , contributing with a kinetic energy $\frac{1}{2m_j} \langle p_{s_j}, p_{s_j} \rangle$, where the bracket denotes the induced inner product in T^*S via the Legendre transform of g . The dynamics in $T^*(S \times \dots \times S)$ is described by the Hamiltonian system*

$$H = \sum \frac{1}{2m_j} \langle p_{s_j}, p_{s_j} \rangle + \sum_{i < j} \kappa_i \kappa_j \hat{G}(s_i, s_j) \quad , \quad \Omega_{\text{collective}} = \Omega_{\text{can}} + \sum_j \kappa_j \Omega(s_j) \quad (3.10)$$

where $\Omega_{\text{can}} = " \sum dp_j \wedge ds_j "$ is the canonical 2-form of $T^*(S \times \dots \times S)$.

We now address the following question. Let $\tilde{g} = h^2 g$ a conformal metric. How do the vortex problems on (S, g) and (S, \tilde{g}) compare? Of course, the symplectic form deforms as $\Omega \rightarrow h^2 \Omega$ in each coordinate. Can we find the new Hamiltonian in terms of the old one? We need transformation formulas both the Green and Robin functions.

3.3 How Green and Robin functions change under conformal transformations

Let $\tilde{g} = h^2 g$ a conformal change of metric. Green's function $G_{\tilde{g}}(s, s_o)$ for the Laplace-Beltrami operator on a closed surface is *not* conformally invariant. It *must* change, because a background uniform vorticity is an area dependent notion, except when it is zero. In fact, if the background vorticity is zero, the following “naive conformal rule” holds:

Proposition 2 *In a zero vorticity background, in order to find the \tilde{g} -regularized stream function for a vortex, subtract $\log(h)/2\pi$ from its g -regularized stream function, $\tilde{g} = h^2 g$.*

Proof.

$$\begin{aligned} \psi_{\text{reg}}(s_o) &= \lim_{s \rightarrow s_o} \psi_{\text{conformal}}(s) - \log d(s, s_o)/2\pi . \\ \log(\tilde{d}(s, s_o)) &= \log \left(d(s, s_o) \frac{\tilde{d}(s, s_o)}{d(s, s_o)} \right) \sim \log(d(s, s_o)) + \log h(s_o) \quad \text{for } s \text{ near } s_o. \end{aligned}$$

However, *if, and this is a big if* the sum of vorticities does not vanish, there is a constant *nonzero* background counteracting vorticity in $S - \{s_1, \dots, s_n\}$. Since what is constant is a metric dependent notion, this complicates the conformal transformation rules. Using Green's function for the Laplace-Beltrami operator on a closed surface becomes unavoidable. The correct transformation formulas are as follows

Lemma 1 ([50])

$$\tilde{G}(s, s_o) - G(s, s_o) = -\frac{1}{A} (\Delta_g^{-1} h^2(s) + \Delta_g^{-1} h^2(s_o)) + \frac{1}{A^2} \int_S h^2 \Delta_g^{-1} h^2 \Omega \quad (3.11)$$

$$\tilde{R}(s) = R(s) - \frac{1}{2\pi} h - \frac{2}{A} \Delta_g^{-1} h^2(s) + \frac{1}{A} \int_S h^2 \Delta_g^{-1} h^2 \Omega \quad (3.12)$$

A beautiful theory has been recently developed around Robins's function and spectral invariants of conformal classes of metrics, see [59]. For our purposes the last term in both formulas can be dropped, as they just add constants. After some algebra juggling, one gets

Theorem 2 *Under a conformal change of metric $\tilde{g}(s) = h^2 g(s)$, the symplectic form deforms as $\Omega \rightarrow h^2 \Omega$ in each coordinate, and the Hamiltonian becomes*

$$\tilde{H} = H - \frac{1}{4\pi} \sum_{\ell=1}^N \kappa_\ell^2 \log(h(s_\ell)) - \frac{\kappa}{A} \sum_{\ell=1}^N \kappa_\ell \Delta^{-1} h^2(s_\ell) \quad , \quad \kappa = \sum_{\ell=1}^N \kappa_\ell . \quad (3.13)$$

Remark 4 *A map $F : (S, g) \rightarrow (\tilde{S}, \tilde{g})$ is conformal when*

$$\tilde{g}(dF_s \cdot v_s, dF_s \cdot w_s) = h^2(s) g(v_s, w_s) . \quad (3.14)$$

The left hand side induces a metric $\tilde{g}_F(v_s, w_s)$ in S , hence $\tilde{g}_F = h^2 g$ and Theorem 2 applies. The vortex dynamics $(\tilde{\Omega}, \tilde{H})$ on $S \times \dots \times S$ corresponds via F to the dynamics of vortices in $\tilde{S} \times \dots \times \tilde{S}$.

For completeness, we present the Proof of lemma 1, following [51]. The trick is average over $\tilde{\Omega}$ twice using the “axioms” in Definition 1. Consider the functions

$$G(s, s_o) - G(s, s_1) = E(s; s_o, s_1) , \quad \tilde{G}(s, s_o) - \tilde{G}(s, s_1) = \tilde{E}(s; s_o, s_1) .$$

Both \tilde{E} and E are harmonic up to (+) and (-) log singularities at s_o and s_1 so they differ by a constant c . To find this constant we do the $\Omega(s) = h^2(s)\Omega(s)$ average of

$$\tilde{E} - E = (\tilde{G}(s, s_o) - \tilde{G}(s, s_1)) - (G(s, s_o) - G(s, s_1)) = c .$$

The first two terms drop out while the last two give

$$\Delta^{-1}h^2(s_1) - \Delta^{-1}h^2(s_o) = c \tilde{A} .$$

Hence

$$(\tilde{G}(s, s_o) - \tilde{G}(s, s_1)) - (G(s, s_o) - G(s, s_1)) = \Delta^{-1}h^2(s_1)/\tilde{A} - \Delta^{-1}h^2(s_o)/\tilde{A} .$$

Again, average this expression, but now over $\Omega(s_1)$. We get

$$\tilde{G}(s, s_o)\tilde{A} - 0 - G(s, s_o)\tilde{A} + \Delta^{-1}h^2(s) = \int_S \Delta^{-1}h^2(s_1)h^2(s_1)\Omega(s_1)/\tilde{A} - \Delta^{-1}h^2(s_o) .$$

This gives the transformation formulas $G \rightarrow \tilde{G}$. The transformation formula $R \rightarrow \tilde{R}$ follows by the limit $s \rightarrow s_o$ in

$$\tilde{G}(s, s_o) - \frac{\log(\tilde{d}(s, s_o))}{2\pi} = (\tilde{G}(s, s_o) - G(s, s_o)) + \left(G(s, s_o) - \frac{\log d(s, s_o)}{2\pi} \right) - \frac{\log(\tilde{d}(s, s_o)/d(s, s_o))}{2\pi} .$$

3.4 Jordan domains, Schottky doubles

The “naive rule” of Proposition 2 works in the case of a Jordan domains $D \subset S$, regardless of the vorticity sum. The basic ingredients for vortex motion are the *hydrodynamical Green functions*:

Definition 2 *A hydrodynamical Green function $G_D(s; s_o)$ is a real harmonic function of $p \in D - s_o$, with (-) logarithmical singularity at $s = s_o$, symmetric in s, s_o , continuous in $\overline{D} - s_o$. G is constant on the boundaries and with desired circulations around them (topology imposes relations).*

C.C. Lin’s result on the added term in the Hamiltonian under a conformal transformation is easily interpreted in terms of Lemma 2, the “naive rule”. Let $f : (D, S, g) \rightarrow (\tilde{D}, \tilde{S}, \tilde{g})$ a conformal map between two Jordan domains,

$$\tilde{g}(df(s) \cdot u, df(s) \cdot v) = h^2(s) g(u, v) . \quad (3.15)$$

Proposition 3 *Let $G_D(s; s_o)$ be the hydrodynamical Green function for D . Then:*

$$\tilde{G}_{\tilde{D}}(\tilde{s}, \tilde{s}_o) = G_D(s; s_o) \quad (3.16)$$

The regularized Green function receives the naive correction,

$$\tilde{g}(s_o) = g(s_o) - \frac{\log(h)}{2\pi} . \quad (3.17)$$

Note that the sign in equation (4.6) of [28] must be corrected from + to -.

For domains conformal to a planar region Green functions are classified into two types. *First type*, more employed in electrostatics, means that $G = 0$ on all boundaries, possibly with different circulations around each of them. *Second kind*, or *modified* Green functions are more commonly used in hydrodynamics. One prescribes zero circulation around the boundaries, except at one of them; the values of $G_D = \text{const.}$ on the boundaries may differ, one of them being normalized to 0. The existence of such Green functions for multiply connected planar domains is due to Koebe, and was used by C.C.Lin in his seminal vortex paper [45].

Specially exciting is the fact that recently Crowdy and Marshall ([18],[19],[21], [22]) brought to “implementational fruition” the task of obtaining Green functions in canonical domains. We believe hydrodynamical Green functions should exist on arbitrary Jordan domains on Riemann surfaces - the difference from planar regions is that in general the domain will have “handles” (like a juice jar).

Remark 5 *An interesting viewpoint is to consider the Schottky double $S_D = D + \overline{D}$ of a Jordan domain D , S is called a symmetric Riemann surface. Consider a metric on S such that the reflection is an isometry. If at initial time each vortex on D corresponds to a symmetric one in \overline{D} , with opposite vorticity, then the corresponding vortices remain symmetric for all time. N -vortex motion on D is the restriction of the corresponding $2N$ vortex motion on S (using the corresponding Hamiltonian from I). Thus Schottky double construction is reminiscent of Thomson’s image vortex method. In spite of the fact that a given metric in D in general will not extend to an reflection-isometric one, since the total vorticity is zero, only the Riemann surface structure is needed. In fact, Schottky-Klein prime functions have been proved useful to finding hydrodynamical Green functions on multiply connected planar domains ([20]).*

4 When the total vorticity vanishes

Recall that when the sum of vorticities does not vanish, using Green’s function for the Laplace-Beltrami operator on a closed surface is unavoidable, because there is a constant *nonzero* background counteracting vorticity in $S - \{s_1, \dots, s_n\}$, and this complicates the conformal transformation rules.

4.1 Riemann surface Green functions

The case where $\sum \kappa_j = 0$ has a special feature: *the collective vortex stream function is the same for all conformal metrics*. Of course, this is the basis of conformal mapping methods in applied hydrodynamics. Mathematically, this follows from the special two-dimensional property

$$\tilde{g} = h^2 g \Rightarrow \Delta_{\tilde{g}} = h^{-2} \Delta_g .$$

Hence, irrespective of the chosen metric in its conformal class, all Laplace-Beltrami operators annihilate the same functions, logarithmic singularities allowed. We need only the complex structure \mathcal{A} : residues at poles are conformal invariants, etc.

Let $G_S = G(s; s_o, s_1)$ be the *two point Green function* for (S, \mathcal{A}) , whose existence for closed Riemann surfaces S can be used as the starting point³ for the theory of meromorphic functions and differentials and uniformization theorems, see e.g. Weyl ([62], II.13, potential arising from a doublet source) or [58]. G is the unique (up to a constant) real *harmonic* function for $s \in S$, except for a (+) logarithmical singularity at s_o and a (-) logarithmical singularity at s_1 .

$\sum_{i=1}^N \kappa_i = 0$ implies that the collective vortex stream function can be given in a *conformally invariant* way:

$$\psi(s) = \psi_{\text{conformal}}(s) = \sum_{j=1}^N \kappa_j G(s; s_j, s^*) \quad (4.1)$$

³It is possible that Riemann [55] and Klein [38] insights came from attended Helmholtz lectures.

where s^* is any chosen point in S . Irrespective of the metric, the background vorticity of every term in $zero$ except at the poles s_j, s^* . Moreover, the choice of s^* is irrelevant as the infinite vorticities there cancel out because $\sum_{i=1}^N \kappa_i = 0$.

[Unfortunately this is not always a free lunch, since there are some drawbacks. One is that Riemann surface two point Green functions have no symmetries in its arguments, which makes it impossible to use $G(s; s_j, s^*)$ for a collective Hamiltonian formulation. Secondly, there is no direct interpretation for the desingularization procedure in terms of invariant objects such as Robin's function. One can also add a practical objection, that it seems to be equally hard finding (in practice) $G_{S,g}(s, s_o)$ or $G_S(s; s_o, s_1)$. In fact, $G_S(s; s_o, s_1) = G_{S,g}(s, s_o) - G_{S,g}(s, s_1) + \text{const.}$ where in the right hand side any metric can be used.]

At any rate, for any vortex problem in which the background vorticity is zero, the “naive conformal rule” (Proposition 2) holds, and this fact may simplify matters considerably.

4.2 Surfaces conformal to S^2

Take the unit sphere $S^2 \subset \mathbb{R}^3$ as the “concrete model” for S (thought of as an abstract Riemann surface). Any metric can be written as $g = h^2 g_o$, where g_o is the canonical metric on the sphere. Stereographic projection $F : S^2 \rightarrow \mathbb{C}$ from the north pole $s^* = (0, 0, 1)$ to the equatorial z -plane yields [35]

$$g_o = h_o^2 |dz|^2, \quad h_o = \frac{2}{1 + |z|^2}. \quad (4.2)$$

We can use the z -plane to study the dynamics. If a particle passes through the north pole, in \mathbb{C} it will disappear in the infinite but reappear instantaneously from another direction, but we may assume that this is not happening during the time frame we are working with⁴. In z -plane, the Green function is $\log(z)/2\pi$. Let $h \cdot h_o$ the combined conformal factor from (S^2, g) to \mathbb{C} . Because of zero total vorticity, vortex dynamics on (S^2, g) can be studied in \mathbb{C} with the symplectic form

$$\Omega = \sum_j \kappa_j h(x_j, y_j)^2 h_o(x_j, y_j)^2 dx_j \wedge dy_j, \quad (4.3)$$

and Hamiltonian given by Theorem 3.13 (we will omit the factor 2π):

$$H = \frac{1}{4} \sum_{j, \ell} \kappa_j \kappa_\ell \log(|z_j - z_\ell|^2) - \sum_{p=1}^N \frac{1}{2} \kappa_p^2 \log(h_o(x_p, y_p) h(x_p, y_p)). \quad (4.4)$$

Here $\sum_{j, \ell}$ means that the $j = \ell$ terms are omitted in the summation. Now we rewrite each κ_m^2 as $-\kappa_m \left(\sum_{p, p \neq m} \kappa_p \right)$, expand and regroup terms. We also use the following interesting formula:

Lemma 2 ([16])

$$|s_1 - s_2|^2 = h_o(s_1) h_o(s_2) |z_1 - z_2|^2. \quad (4.5)$$

Here the distances are the euclidian distances in \mathbb{R}^2 and \mathbb{R}^3 .

Pulling back to the sphere, we get:

Proposition 4 *For a system of N point vortices in the sphere S^2 with a metric $g = h^2 g_o$ (g_o is the standard metric), and such that the total vorticity vanishes, the dynamics is governed by*

$$\Omega = \sum_j \kappa_j h^2(s_j) \omega(s_j), \quad H = \frac{1}{4} \sum_{j, \ell} \kappa_j \kappa_\ell \log(h(s_j) h(s_\ell) |s_j - s_\ell|^2) \quad (4.6)$$

where $|s_j - s_\ell|$ is the euclidian distance and ω is the area form of the sphere.

⁴*Esse est percipi*, stealing the motto of Bishop Berkeley, ironically the greatest foe of Calculus lovers.

4.3 Proof of Kimura's conjecture on dipole motion

The Hamiltonian for a vortex pair with opposite vorticities can be written as

$$H = -\kappa^2 \frac{\log d(s_1, s_2)}{2\pi} + \kappa^2 B(s_1, s_2), \quad B(s_1, s_2) = \left[\frac{R(s_1) + R(s_2)}{2} - \left(G(s_1, s_2) - \frac{\log d(s_1, s_2)}{2\pi} \right) \right]. \quad (4.7)$$

Let $\kappa = O(\epsilon)$ and initial conditions $d(s_1(0), s_2(0)) = O(\epsilon)$. Kimura's conjectured that as $\epsilon \rightarrow 0$ the vorticity center moves on a geodesic. The truncated system taking only the first term in (4.7) yields a system of ODEs on $D(s_o) \times D(s_o)$, the product of two copies of a geodesic ball centered on $s_o \in S$, given by

$$\dot{s}_1 = -\kappa \operatorname{sgrad}_{s_1} \log d(s_1, s_2), \quad \dot{s}_2 = \kappa \operatorname{sgrad}_{s_2} \log d(s_1, s_2). \quad (4.8)$$

We claim that this equation represents the dominant $O(1)$ geodesic dynamics plus an $O(\epsilon^2)$ perturbation. Due to its symmetric form, $B(s_1, s_2)$ also produces an $O(\epsilon^2)$ perturbation. We take Gauss coordinates ([60], [23])

$$ds^2 = du^2 + G(u, v)dv^2, \quad G(0, v) = 1, \quad \frac{\partial}{\partial u}|_{u=0} G(u, v) = 0. \quad (4.9)$$

The u -curves form a field of geodesics, and the central v -curve (for $u = 0$) is also a geodesic. Suppose s_o corresponds to some v and $u = 0$. In these coordinates, $s_1(0) = (-\epsilon, v)$ and $s_2(0) = (\epsilon, v)$. Clearly, at $t = 0$, $\dot{s}_{1,2}$ will be tangent to v -curves, and in the Gauss coordinates,

$$\dot{v}_1(0) = \kappa/2\epsilon \frac{1}{\sqrt{G(-\epsilon, v)}}, \quad \dot{v}_2(0) = \kappa/2\epsilon \frac{1}{\sqrt{G(\epsilon, v)}}, \quad \dot{u}_1(0) = \dot{u}_2(0) = 0. \quad (4.10)$$

In the limit as $\epsilon \rightarrow 0$ (4.9), with $\kappa = 2\kappa_o\epsilon$, $\kappa_o = 1$, we get

$$\dot{v}_1(0) = \dot{v}_2(0) = 1 + O(\epsilon), \quad \dot{u}_1(0), \dot{u}_2(0) = O(\epsilon). \quad (4.11)$$

Note that $\frac{\partial}{\partial u}|_{u=0} G(u, v) = 0$ implies that (4.8) does not have an $O(\epsilon)$ term. This concludes the proof.

5 Final Remarks

5.1 Robin's function and vortex drift

A single vortex $\dot{s}_o = \operatorname{sgrad} R(s_o)$ describes contour lines of Robin's function, so it would be interesting to understand its generic properties. For metrics on the sphere R satisfies (see also (3.6), [59])

$$\Delta R = h^2(K_E - 4\pi/A(E)) \quad (5.1)$$

where Δ is the standard Laplacian on the sphere.

5.2 Batman's function and integrable vortex pair problems on Liouville surfaces

If S embeds in \mathbf{R}^3 having an axis of symmetry n , the momentum map of the S^1 action on $S \times \dots \times S$ with $\Omega = \sum \kappa_j \Omega_{s_j}$ is $J = (\sum \kappa_j s_j) \cdot n$. In particular, a vortex pair on a surface of revolution is integrable, with second integral $f = (s_1 - s_2) \cdot n$. Can we find nontrivial examples of integrable vortex pairs? As it is well known, Jacobi has shown that the geodesic flow on the ellipsoid is integrable. In view of Kimura's theorem, vortex pairs extends geodesic problems, so the following question is natural:

Question 1 *Is the vortex pair problem on the triaxial ellipsoid (with opposite vorticities) integrable? If it is not, can one prove it analytically by Melnikov's method applied to the geodesics passing by the umbilics?*

Conformal maps from the triaxial ellipsoid to the standard sphere are known ([17], [47], [56]). The conformal factor h^2 can be retrieved from these papers, so Proposition 4 can be implemented. Alternatively, using sphero-conical coordinates (see, eg. [39]) allows obtaining h^2 from scratch with two elliptic integrals. More generally, Liouville surfaces (see e.g., [9]) are those for which the geodesic flow is integrable. For which Liouville surfaces is the vortex pair system integrable? In this programme we have found some preliminary results [41] for (4.7) showing how the geodesic system is perturbed for $d(s_1, s_2) = O(\epsilon)$. The $O(\epsilon^2)$ perturbation term is

$$B(s_1, s_2) = \left[\frac{R(s_1) + R(s_2)}{2} - \left(G(s_1, s_2) - \frac{\log d(s_1, s_2)}{2\pi} \right) \right]$$

which could be called, for the sake of justice, Batman's function.

5.3 Time dependent problems

Either on closed surfaces or Jordan domains it is natural to consider time dependent problems, subject to the natural constraint (in view of incompressibility) that the total area with respect to the metrics $g(t)$ does not change. For the sphere and disks Riemann mapping theorem guarantees that the change is always conformal. In the general case the complex structure may also change, an interesting complication that should be addressed.

5.4 Higher dimensional extensions

Alan Weinstein called our attention that the statement of Theorem 1 makes sense on a compact Kahler manifold, replacing the two dimensional objects by their higher dimensional analogues. In particular, Calabi-Yau manifolds are very fashionable objects nowadays. Would the generalization of (3.7, 3.8) represent singular solutions to Euler's equation for the hydrodynamics on (S, g) , i.e. the geodesic flow on $\text{SDiff}(S)$, in the spirit of [3] and [46]? Or will that construction be more in line with field theorists approaches, see e.g., Baez [4]?

5.5 Vortices for a proof of Riemann hypothesis?

One of the approaches for a proof of Riemann hypothesis is searching a connection between the nontrivial zeros of Riemann's zeta function with some quantum mechanical problem (Berry and Keating, [5]). The quantum version of geodesic motion on (S, g) , i.e., the free particle, leads to the spectrum of minus the Laplace-Beltrami operator. As a vortex dipole moves on geodesics, why not look for a proof quantizing a vortex problem, which is asymmetric with respect to time reversal?

"Sir Michael Berry proposes that there exists a classical dynamical system, asymmetric with respect to time reversal, the lengths of whose periodic orbits correspond to the rational primes, and whose quantum-mechanical analog has a Hamiltonian with zeros equal to the imaginary parts of the nontrivial zeros of the zeta function. The search for such a dynamical system is one approach to proving the Riemann hypothesis (Daniel Bump, [14])."

We conclude this note on a highly speculative tone:

Conjecture 1 *A connection may exist between the zeros of Riemann's zeta function and the quantization of a 3/2 degrees of freedom vortex monopole problem on some compact Riemann surface with metric g ,*

$$\dot{s} = \text{sgrad } G(s, s_o) \quad , \quad \dot{s}_o = \text{sgrad } R(s_o) \quad ,$$

or with a vortex pair

$$\dot{s}_1 = -\kappa \text{sgrad}_{s_1} H(s_1, s_2) \quad \dot{s}_2 = \kappa \text{sgrad}_{s_2} H(s_1, s_2) \quad , \quad H = -\kappa^2 \frac{\log d(s_1, s_2)}{2\pi} + \kappa^2 B(s_1, s_2) \quad .$$

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